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Bases for representations of quantum algebras

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Abstract. We derive an explicit expression for the eigenfunctions and the corresponding eigenvalues of the operator $[q^{1/4}J_+(q) + q^{-1/4}J_-(q)]q^{J_3(q)/2}$ in an arbitrary irreducible representation of the algebra $su_q(2)$. The general form of the intertwining operator $A^J(q)$, which is a q-extension of the classical $su(2)$-operator a^J, $J_1a^J = a^J J_3$, is also found. The matrix elements of $A^J(q)$ are expressed in terms of the dual q-Kravchuk polynomials.

1. Introduction

The purpose of this paper is to introduce and study a 'nonstandard' basis for the irreducible representations of the quantum group $su_q(2)$. A 'standard' basis was introduced and studied elsewhere [1–4].

Let us first of all explain the meaning of the words 'standard' and 'nonstandard' in the above paragraph. For a Lie algebra L, or Lie group G, we obtain a 'standard' basis by taking a chain of subalgebras $L \supset L_1 \supset L_2 \supset \cdots \supset L_N$ and writing down the Casimir operators of all the algebras in the chain. The basis functions are then constructed as simultaneous eigenfunctions of all these Casimir operators (and possibly some further commuting operators). For the Lie algebra $su(2)$ with basis J_1, J_2, and J_3, there is, up to the equivalence, just one standard, or canonical, basis, corresponding to the subgroup chain $SU(2) \supset U(1)$. The basis functions are eigenfunctions of $J^2 = J_1^2 + J_2^2 + J_3^2$ and J_3, i.e.

$$J^2\psi_{Jm} = J(J+1)\psi_{Jm} \qquad J_3\psi_{Jm} = m\psi_{Jm} \tag{1.1}$$

where $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, -J \leqslant m \leqslant J$. These functions can be realized as monomials z^{J+m}, or as spherical harmonics $Y_{J,m}(\theta, \phi)$ on the sphere S_2 [5].

Nonstandard bases for Lie groups and Lie algebras can be obtained by diagonalizing other complete sets of commuting operators. For $su(2)$ nothing new is obtained by replacing J_3 by a general element $aJ_1 + bJ_2 + cJ_3$ of the Lie algebra. New, nonstandard, bases are obtained by diagonalizing higher-order polynomials in the enveloping algebra of $su(2)$. For instance, any second-order operator $a_{ik}J_iJ_k$ can be rotated into $a_1J_1^2 + a_2J_2^2 + a_3J_3^2$. If we take common eigenfunctions of the operators Q and J^2, i.e.

$$\begin{aligned} &Q\psi_{Jk} = k\psi_{Jk} \qquad J^2\psi_{Jk} = J(J+1)\psi_{Jk} \\ &Q = J_1^2 + pJ_2^2 \qquad 0 < p < 1 \end{aligned} \tag{1.2}$$

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we obtain a new basis. When realized on the sphere S_2, the basis functions are products of Lamé polynomials. They can be obtained by separating variables in the Laplace equation on the sphere S_2 in elliptic coordinates [6].

The motivation for studying different bases for representations of Lie groups and Lie algebras is multifold. A mathematical motivation arises, for instance, from the theory of special functions and in particular orthogonal polynomials [7, 8]. Indeed, different bases for the same representations of the Lie group lead to different special functions and provide a group theoretical underpinning for all of these functions.

A complementary motivation for studying different bases comes from physics. In particular, consider the group $SU(2)$, or equivalently $O(3)$. The nonstandard basis (1.2) corresponds to the subgroup chain $O(3) \supset D_2$, where D_2 is the dihedral group, generated by rotations through π about each of the coordinate axes in the euclidean space E_3. This group–subgroup chain and the corresponding basis functions occur in atomic, nuclear and molecular physics, because of their relation to asymmetric tops [9]. More generally, nonstandard bases of $O(3)$ occur in molecular physics and quantum chemistry as 'symmetry adapted wave functions' [10]. They correspond to subgroup chains $O(3) \supset \Gamma$, where Γ is a discrete (finite) subgroup of $O(3)$.

Now let us turn to the case of quantum algebras, in particular, $su_q(2)$. The quantum algebra may have an ordinary Lie subalgebra L (or chain of subalgebras). This subalgebra can then be used to introduce a 'standard', or 'canonical' basis for the considered quantum algebra. This has been done for $su_q(2)$, using the chain $su_q(2) \supset u(1)$ [1–4]. In particular, basis functions of $su_q(2)$ were realized on an ordinary sphere S_2 as eigenfunctions of the operator $J_3(q)$ (a differential operator) and of the Casimir operator of $su_q(2)$ (a differential–difference operator). These basis functions were expressed in terms of q-Jacobi and q-Legendre polynomials [4].

For quantum algebras the distinction between the 'algebra' and an enveloping algebra is much less rigid than for Lie algebras. Moreover, it is certainly not true that any linear combination of the $su_q(2)$ elements $J_3(q)$, $J_+(q)$ and $J_-(q)$ (see below) can be transformed into $J_3(q)$. Hence many different inequivalent bases for quantum algebras exist. Their study is of interest in q-special function theory. It is also relevant for any physical theory in which a quantum group appears as a symmetry group, or as a dynamic group of some kind.

In this paper we construct one such nonstandard basis for the algebra $su_q(2)$ by diagonalizing a specific operator, studied earlier in a different context by Ballesteros and Chumakov [11]. In section 2 we assemble known facts about overlap functions for $su(2)$ representations, stressing the features to be extended to the case of the quantum algebra $su_q(2)$. In section 3 we derive an explicit expression for the eigenfunctions and the corresponding eigenvalues of the operator $[q^{1/4}J_+(q) + q^{-1/4}J_-(q)]q^{J_3(q)/2}$ in an arbitrary irreducible representation of the algebra $su_q(2)$. The general form of the intertwining operator $A^J(q)$, which is a q-extension of the classical $su(2)$-operator a^J, $J_1 a^J = a^J J_3$, is also found. The matrix elements of $A^J(q)$ are expressed in terms of the dual q-Kravchuk polynomials. The concluding section 4 indicates some further research directions of interest. Finally, the appendix contains those properties of the Kravchuk and q-Kravchuk polynomials which provide the background for our discussion. We shall assume throughout the paper that the deformation parameter q belongs to the interval (0, 1), although there seems to be no difficulty in extending our results to the case when $0 < |q| < 1$ (see in this connection [12]).

2. Overlap functions for $su(2)$ representations

The commutation relations for the Hermitian generators J_1, J_2 and J_3 of the $su(2)$ algebra are

$$[J_1, J_2] = \mathrm{i}J_3 \qquad [J_2, J_3] = \mathrm{i}J_1 \qquad [J_3, J_1] = \mathrm{i}J_2. \tag{2.1}$$

It is well known [5] that for any irreducible representation with $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, the generators J_3 and $J_\pm = J_1 \pm \mathrm{i}J_2$ define an orthogonal basis consisting of the normalized eigenvectors of J_3 by the equations

$$J_3 f_m^J = m f_m^J \qquad J_\pm f_m^J = [(J \pm m + 1)(J \mp m)]^{1/2} f_{m\pm1}^J. \tag{2.2}$$

The raising J_+ and lowering J_- operators satisfy the commutation relations

$$[J_+(x), J_-(x)] = 2J_3(x) \qquad [J_3(x), J_\pm(x)] = \pm J_\pm(x) \tag{2.3}$$

which follow from (2.1).

If one uses the similarity transform

$$\mathrm{e}^{\mathrm{i}\alpha A} B \mathrm{e}^{-\mathrm{i}\alpha A} = B + \sum_{n=1}^{\infty} \frac{(\mathrm{i}\alpha)^n}{n!} \underbrace{[A, \ldots, [A, [A, B]]]}_{n\text{-fold}} \tag{2.4}$$

with $A = J_2$ and $B = J_1$, then from the commutation relations (2.1) it follows that

$$\mathrm{e}^{\mathrm{i}\alpha J_2} J_1 \mathrm{e}^{-\mathrm{i}\alpha J_2} = \cos\alpha J_1 + \sin\alpha J_3. \tag{2.5}$$

When $\alpha = \pi/2$, (2.5) reduces to

$$\mathrm{e}^{\pi \mathrm{i} J_2/2} J_1 \mathrm{e}^{-\pi \mathrm{i} J_2/2} = J_3. \tag{2.5$'$}$$

Later on we shall need (2.5′), written in the intertwining form

$$J_1 \mathrm{e}^{-\pi \mathrm{i} J_2/2} = \mathrm{e}^{-\pi \mathrm{i} J_2/2} J_3. \tag{2.5$''$}$$

In the basis consisting of the eigenfunctions f_m^J, $-J \leqslant m \leqslant J$, the generator J_3 is diagonal. The relation (2.5″) defines the basis $\tilde{f}_m^J$, in which the generator J_1 is diagonal. Indeed, since

$$J_1 \mathrm{e}^{-\pi \mathrm{i} J_2/2} f_m^J = \mathrm{e}^{-\pi \mathrm{i} J_2/2} J_3 f_m^J = m \mathrm{e}^{-\pi \mathrm{i} J_2/2} f_m^J \tag{2.6}$$

it becomes evident that

$$\tilde{f}_m^J = \mathrm{e}^{-\pi \mathrm{i} J_2/2} f_m^J. \tag{2.6$'$}$$

It is clear that the operator $\exp(-\pi \mathrm{i} J_2/2)$ is not the only one that relates J_1 to J_3 (see (2.5″)). Indeed, if some operator B intertwines operators A and C, i.e. $AB = BC$, then any other operator of the form $f_1(A)Bf_2(C)$, where f_1 and f_2 are arbitrary analytic functions, will possess the same property. For instance, two more operators, satisfying the same relation (2.5″), are obtained in the following way. From (2.3) and the similarity transform (2.4) with $A = J_\pm$ and $B = J_3$ it follows that

$$\mathrm{e}^{\mathrm{i}\alpha J_\pm} J_3 \mathrm{e}^{-\mathrm{i}\alpha J_\pm} = J_3 \mp \mathrm{i}\alpha J_\pm \tag{2.7}$$

respectively. Similarly, it is easy to show that

$$\mathrm{e}^{\mathrm{i}\alpha J_\pm} J_\mp \mathrm{e}^{-\mathrm{i}\alpha J_\pm} = J_\mp \pm 2\mathrm{i}\alpha J_3 + \alpha^2 J_\pm. \tag{2.8}$$

Now, if one uses the first relation from (2.7) with $\alpha = \mathrm{i}/2$ and then the second one from (2.8) with $\alpha = -\mathrm{i}$, this yields

$$J_1 \mathrm{e}^{J_-} \mathrm{e}^{-J_+/2} = \mathrm{e}^{J_-} \mathrm{e}^{-J_+/2} J_3. \tag{2.9}$$

In a like manner, from the second relation in (2.7) with $\alpha = -\mathrm{i}/2$ and the first one from (2.8) with $\alpha = \mathrm{i}$ it follows that

$$J_1 \mathrm{e}^{-J_+} \mathrm{e}^{J_-/2} = \mathrm{e}^{-J_+} \mathrm{e}^{J_-/2} J_3. \tag{2.10}$$

It is not hard to find the general form of the operator a^J (for each irreducible representation of $su(2)$, characterized by some integer or half-integer number J), satisfying the relation

$$J_1 a^J = a^J J_3. \tag{2.11}$$

Matrix elements of the operators J_3 and $J_\pm$ in the canonical basis (2.2) are

$$(J_3)_{m,m'} = m\delta_{m,m'} \qquad (J_\pm)_{m,m'} = [(J \pm m)(J \mp m + 1)]^{1/2}\delta_{m,m'\pm 1}. \tag{2.12}$$

Taking into account that $J_1 = (J_+ + J_-)/2$, and using (2.12), leads to the following matrix form of equation (2.11):

$$\sqrt{(J+m+1)(J-m)}a^J_{m+1,m'} + \sqrt{(J-m+1)(J+m)}a^J_{m-1,m'} = 2m'a^J_{m,m'}. \tag{2.13}$$

To dispense with the square roots in (2.13), substitute

$$a^J_{m,m'} = \begin{bmatrix} 2J \\ J+m \end{bmatrix}^{1/2} b^J_{m,m'} \tag{2.14}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ are the binomial coefficients,

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{n!}{k!(n-k)!} = \frac{k+1}{n-k}\begin{bmatrix} n \\ k+1 \end{bmatrix} = \frac{n-k+1}{k}\begin{bmatrix} n \\ k-1 \end{bmatrix}. \tag{2.15}$$

This results in the simple recurrence relation

$$(J-m)b^J_{m+1,m'} + (J+m)b^J_{m-1,m'} = 2m'b^J_{m,m'} \tag{2.13$'$}$$

for the matrix elements $b^J_{m,m'}$, which is a particular case of the recurrence relation for the Kravchuk polynomials $K_n(x; p, N)$ with the parameter $p = \frac{1}{2}$ (see (A.2)). Consequently,

$$b^J_{m,m'} = \chi^J(m')K_{J+m}(J-m'; \tfrac{1}{2}, 2J) \tag{2.16}$$

where $\chi^J(m')$ is some arbitrary function of m'. Substituting (2.16) into the right-hand side of (2.14) gives

$$a^J_{m,m'} = \chi^J(m')\begin{bmatrix} 2J \\ J+m \end{bmatrix}^{1/2} K_{J+m}(J-m'; \tfrac{1}{2}, 2J). \tag{2.17}$$

Since $K_0(x; p, N) = 1$ by the initial condition, one can also represent (2.17) as

$$a^J_{m,m'} = a^J_{-J,m'}\begin{bmatrix} 2J \\ J+m \end{bmatrix}^{1/2} K_{J+m}(J-m'; \tfrac{1}{2}, 2J). \tag{2.17$'$}$$

Observe that in the particular case when $a^J = \exp(-\pi \mathrm{i} J_2/2)$ (cf (2.5″) and (2.6′)), the matrix elements $a^J_{m,m'}$ coincide with Wigner's d-function (see, for example, [8]):

$$d^J_{m,m'}\left(\frac{\pi}{2}\right) = \frac{1}{2^J}\begin{bmatrix} 2J \\ J+m \end{bmatrix}^{1/2}\begin{bmatrix} 2J \\ J+m' \end{bmatrix}^{1/2} K_{J+m}\left(J-m'; \frac{1}{2}, 2J\right). \tag{2.18}$$

In what follows we shall consider a particular realization for the generators J_3, $J_\pm$ and, consequently, for the eigenfunctions f^J_m. If one defines (cf [13, p 278]),

$$J_+(x) = x\left(2J - x\frac{\mathrm{d}}{\mathrm{d}x}\right) \qquad J_-(x) = \frac{\mathrm{d}}{\mathrm{d}x} \qquad J_3(x) = x\frac{\mathrm{d}}{\mathrm{d}x} - J \tag{2.19}$$

then it is easy to verify that $J_\pm(x)$ and $J_3(x)$ satisfy the commutation relations (2.3). The eigenfunctions of the operator $J_3(x)$ are monomials

$$f^J_m(x) = c^J_m x^{J+m} \qquad c^J_m = \begin{bmatrix} 2J \\ J+m \end{bmatrix}^{1/2}. \tag{2.20}$$

One can easily check that the basis $f_m^J(x)$ is canonical, i.e. equations (2.2) are satisfied.

For the realization (2.19) the eigenfunctions of the generator $J_1(x) = \frac{1}{2}[J_+(x) + J_-(x)] = Jx + \frac{1}{2}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}$ have the form

$$\varphi_m^J(x) := (1+x)^{J+m}(1-x)^{J-m} = \varphi_{-m}^J(-x). \tag{2.21}$$

In agreement with (2.4), the eigenvalues of $J_1(x)$ are the same as of $J_3(x)$, i.e.

$$J_1(x)\varphi_m^J(x) = m\varphi_m^J(x). \tag{2.22}$$

To expand $\varphi_m^J(x)$ in terms of $f_m^J(x)$ one can use a generating function for the Kravchuk polynomials $K_n(x; p, N)$ (see (A.5′)). Taking into account the normalization constant c_m^J in (2.20), the relation (A.5′) is equivalent to the expansion

$$\varphi_m^J(x) = \sum_{m'=-J}^{J} \alpha_{m',m}^J f_{m'}^J(x) \tag{2.23}$$

where the connection coefficients $\alpha_{m',m}^J$ are equal to

$$\alpha_{m',m}^J = \begin{bmatrix} 2J \\ J+m' \end{bmatrix}^{1/2} K_{J+m'}(J-m; \tfrac{1}{2}, 2J). \tag{2.24}$$

If one compares (2.24) with (2.17′), then it becomes evident that the connection coefficients $\alpha_{m,m'}^J$ are the particular case of the matrix elements $a_{m,m'}^J$ of the intertwining operator (2.11) with $a_{-J,m'}^J = 1, -J \leqslant m' \leqslant J$.

3. Representations of the $su_q(2)$ algebra in a nonstandard basis

The commutation relations in this case are [1, 2]

$$[J_1(q), J_2(q)] = \frac{\mathrm{i}}{2}[2J_3(q)]_q \qquad [J_2(q), J_3(q)] = \mathrm{i}J_1(q) \qquad [J_3(q), J_1(q)] = \mathrm{i}J_2(q) \tag{3.1}$$

where

$$[A]_q := \frac{q^{A/2} - q^{-A/2}}{q^{1/2} - q^{-1/2}}. \tag{3.2}$$

In terms of the raising $J_+(q) = J_1(q) + \mathrm{i}J_2(q)$ and lowering $J_-(q) = J_1(q) - \mathrm{i}J_2(q)$ operators, (3.1) take the form

$$[J_+(q), J_-(q)] = [2J_3(q)]_q \qquad [J_3(q), J_\pm(q)] = \pm J_\pm(q). \tag{3.3}$$

The canonical basis in each irreducible representation of the algebra $su_q(2)$, characterized by some integer or half-integer number J, is defined as [3]

$$J_3(q)f_m^J(q) = mf_m^J(q) \qquad J_\pm(q)f_m^J(q) = [J \pm m + 1]_q^{1/2}[J \mp m]_q^{1/2} f_{m\pm1}^J(q). \tag{3.4}$$

As is shown in [11], one can also diagonalize the operator

$$\tilde{J}_1(q) := \tfrac{1}{2}q^{J_3(q)/4}[J_+(q) + J_-(q)]q^{J_3(q)/4} = \tfrac{1}{2}\{q^{1/4}J_+(q) + q^{-1/4}J_-(q)\}q^{J_3(q)/2}. \tag{3.5}$$

Below we give an analytic proof of this fact by finding eigenfunctions and eigenvalues of the operator $\tilde{J}_1(q)$. But we begin this section with a derivation of the explicit form of the operator $A^J(q)$, that intertwines $\tilde{J}_1(q)$ and $[2J_3(q)]_q/2$. The operator $A^J(q)$ is thus defined by the relation

$$2\tilde{J}_1(q)A^J(q) = A^J(q)[2J_3(q)]_q \tag{3.6}$$

and in the limit as $q \to 1^-$ it coincides with a^J in (2.11).

Matrix elements of the operators $J_3(q)$ and $J_\pm(q)$ in the canonical basis (3.4) are

$$(J_3(q))_{m,m'} = m\delta_{m,m'} \qquad (J_\pm(q))_{m,m'} = [J \pm m]_q^{1/2}[J \mp m + 1]_q^{1/2}\delta_{m,m'\pm 1}. \tag{3.7}$$

Employing (3.5) and (3.7) yields the following matrix form of equation (3.6):

$$\begin{aligned} q^{(2m+1)/4}&[J+m+1]_q^{1/2}[J-m]_q^{1/2}A^J_{m+1,m'}(q) \\ &+q^{(2m-1)/4}[J-m+1]_q^{1/2}[J+m]_q^{1/2}A^J_{m-1,m'}(q) \\ &= [2m']_q A^J_{m,m'}(q). \end{aligned} \tag{3.8}$$

To simplify (3.8), substitute

$$A^J_{m,m'}(q) = q^{(m^2-J^2)/4+m(2J-1)/4}\begin{bmatrix} 2J \\ J+m \end{bmatrix}_q^{1/2} B^J_{m,m'}(q) \tag{3.9}$$

where $\begin{bmatrix} m \\ n \end{bmatrix}_q$ is the q-binomial coefficient,

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q;q)_m}{(q;q)_n(q;q)_{m-n}} = q^{n(m-n)/2}\frac{[m]_q!}{[n]_q![m-n]_q!} \tag{3.10}$$

$[m]_q! = \prod_{j=1}^m [j]_q$ and the q-shifted factorial is defined as $(z;q)_0 = 1$ and $(z;q)_n = \prod_{j=0}^{n-1}(1-zq^j), n = 1,2,3,\ldots$ (we employ the standard notations of q-analysis, see [14], or [15]). This results in a recurrence relation with respect to the index m:

$$(1-q^{m-J})B^J_{m+1,m'}(q) - q^{-2J}(1-q^{J+m})B^J_{m-1,m'}(q) = q^{-J}(q^{m'}-q^{-m'})B^J_{m,m'}(q) \tag{3.11}$$

which is a particular case of the three-term recurrence relation for the dual q-Kravchuk polynomials $K_n(x(s);c,N|q)$ with the parameters $n = J+m$, $s = J-m'$, and $N = 2J$. Consequently,

$$A^J_{m,m'}(q) = q^{(m^2-J^2)/4+m(2J-1)/4}\begin{bmatrix} 2J \\ J+m \end{bmatrix}_q^{1/2} K_{J+m}(x(J-m');-1,2J|q)\chi^J(m'|q) \tag{3.12}$$

where $x(J-m') = q^{-J}(q^{m'}-q^{-m'})$ and $\chi^J(m'|q)$ is some arbitrary function of the index m'. Since $K_0(x(s);c,N|q) = 1$ by the initial condition, it is more convenient to represent (3.12) as

$$A^J_{m,m'}(q) = q^{(J+m)(J+m-1)/4}\begin{bmatrix} 2J \\ J+m \end{bmatrix}_q^{1/2} K_{J+m}(x(J-m');-1,2J|q)A^J_{-J,m'}(q). \tag{3.12$'$}$$

As in the case of the $su(2)$ algebra, matrix $A^J_{m,m'}(q)$ depends on the $2J+1$ parameters $A^J_{-J,m'}(q)$, $-J \leqslant m' \leqslant J$, and has the same structure as matrix $a^J_{m,m'}$ in (2.17$'$). In the limit as $q \to 1^-$, $A^J_{m,m'}(q)$ coincides with $a^J_{m,m'}$. Notice also that one can express the matrix elements $A^J_{m,m'}(q)$ in terms of the q-Kravchuk polynomials $K_{J-m'}(q^{-(J+m)};q^{-2J},2J;q)$ on the q-linear lattice q^{-s} as well (see (A.6) and (A.11)).

The next problem is to find the explicit form of eigenfunctions and eigenvalues of the operator $\tilde{J}_1(q)$. To do so we employ the particular realization (cf [16])

$$\begin{aligned} J_+(x;q) &= x\left[2J - x\frac{\mathrm{d}}{\mathrm{d}x}\right]_q = x[J - J_3(x)]_q \\ J_-(x;q) &= \frac{1}{x}\left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]_q = \frac{1}{x}[J + J_3(x)]_q \\ J_3(x;q) &= x\frac{\mathrm{d}}{\mathrm{d}x} - J = J_3(x) \end{aligned} \tag{3.13}$$

for the $su_q(2)$ algebra. It is easy to verify that (3.13) satisfy the commutation relations (3.3); moreover, the canonical basis in this case also consists of monomials in x, i.e.

$$f_m^J(x;q) = c_m^J(q)x^{J+m} \qquad c_m^J(q) = q^{(m^2-J^2)/4}\begin{bmatrix} 2J \\ J+m \end{bmatrix}_q^{1/2}. \tag{3.14}$$

By analogy with the classical case of the $su(2)$ algebra, let us look for eigenfunctions of $\tilde{J}_1(x;q)$ of the form

$$\varphi_m^J(x;q) = (ax;q)_{J-m}(-bx;q)_{J+m} \tag{3.15}$$

where a and b do not depend on x, but may be q dependent. Since

$$q^{cx\frac{\mathrm{d}}{\mathrm{d}x}} f(x) = f(q^c x) \qquad c \in \mathbb{R} \tag{3.16}$$

we have

$$q^{J_3(x)/2}\varphi_m^J(x;q) = q^{-J/2}q^{\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x}}\varphi_m^J(x;q) = q^{-J/2}\varphi_m^J(q^{1/2}x;q). \tag{3.17}$$

By using (3.16), one can now evaluate

$$\begin{aligned} J_+(x;q)q^{J_3(x)/2}\varphi_m^J(x;q) &= q^{-J/2}J_+(x;q)\varphi_m^J(q^{1/2}x;q) \\ &= \frac{q^{-J/2}x}{q^{1/2}-q^{-1/2}}[q^{J-\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x}} - q^{\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x}-J}]\varphi_m^J(q^{1/2}x;q) \\ &= \frac{q^{-J/2}x}{q^{1/2}-q^{-1/2}}[q^J\varphi_m^J(x;q) - q^{-J}\varphi_m^J(qx;q)]. \end{aligned} \tag{3.18}$$

In a like manner,

$$\begin{aligned} J_-(x;q)q^{J_3(x)/2}\varphi_m^J(x;q) &= q^{-J/2}J_-(x;q)\varphi_m^J(q^{1/2}x;q) \\ &= \frac{q^{-J/2}}{x(q^{1/2}-q^{-1/2})}[\varphi_m^J(qx;q) - \varphi_m^J(x;q)]. \end{aligned} \tag{3.19}$$

Since

$$(qz;q)_n = \frac{1-zq^n}{1-z}(z;q)_n$$

by the definition of the symbol $(z;q)_n$, we have

$$\varphi_m^J(qx;q) = \frac{(1-axq^{J-m})(1+bxq^{J+m})}{(1-ax)(1+bx)}\varphi_m^J(x;q). \tag{3.20}$$

In accordance with (3.5), multiply (3.18) by $q^{1/4}$ and (3.19) by $q^{-1/4}$ and then sum them up. After that it becomes clear that $\varphi_m^J(x;q)$ will be eigenfunctions of the operator $\tilde{J}_1(x;q)$ with eigenvalues $[2m]_q/2$ provided that $a = b = q^{1/4-J/2}$. We have thus obtained that

$$\tilde{J}_1(x;q)\varphi_m^J(x;q) = \frac{[2m]_q}{2}\varphi_m^J(x;q) \tag{3.21a}$$

$$\varphi_m^J(x;q) := (q^{1/4-J/2}x;q)_{J-m}(-q^{1/4-J/2}x;q)_{J+m} = \varphi_{-m}^J(-x;q). \tag{3.21b}$$

Clearly, the $\varphi_m^J(x;q)$ are also eigenfunctions of the invariant (i.e. commuting with $J_+(q)$, $J_-(q)$, and $J_3(q)$) Casimir operator

$$C(q) := J_+(q)J_-(q) + [J_3(q) - \tfrac{1}{2}]_q^2 - \tfrac{1}{4} \tag{3.22}$$

of the $su_q(2)$ algebra, with the eigenvalues $[J+\frac{1}{2}]_q^2 - \frac{1}{4}$.

To expand $\varphi_m^J(x;q)$ in terms of the eigenfunctions $f_m^J(x;q)$ of the operator $J_3(x;q)$, one can employ the generating function for the dual q-Kravchuk polynomials (A.14). But we remark first that in the two 'extremal' cases when $m = \pm J$, the functions $\varphi_m^J(x;q)$ have simpler forms (for when $m = J$ the first factor in the definition of $\varphi_m^J(x;q)$ is equal to unity, whereas when $m = -J$, the second factor becomes unity):

$$\varphi_J^J(x;q) = (-q^{1/4-J/2}x;q)_{2J} = \varphi_{-J}^J(-x;q). \tag{3.23}$$

Therefore in these two cases the relation between $\varphi_m^J(x;q)$ and $f_m^J(x;q)$ is straightforward. Indeed, with the aid of the identity

$$(z;q)_n = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q (-z)^k \tag{3.24}$$

one can expand $\varphi_J^J(x;q)$ in powers of x, i.e.

$$\varphi_J^J(x;q) = \sum_{k=0}^{2J} q^{k(k-J-1/2)/2} \begin{bmatrix} 2J \\ k \end{bmatrix}_q x^k. \tag{3.25}$$

Taking into account the explicit form of the normalization constant $c_m^J(q)$ in (3.14), relation (3.25) is equivalent to the following expansion:

$$\varphi_J^J(x;q) = \sum_{m=-J}^{J} q^{(J+m)(J+m-1)/4} \begin{bmatrix} 2J \\ J+m \end{bmatrix}_q^{1/2} f_m^J(x;q). \tag{3.26}$$

Later on we shall use (3.23) and (3.26) for a consistency check of expansions in the general case of arbitrary values of $m \in \{-J, -J+1, \ldots, J\}$.

Now substituting $k = J - m$, $N = 2J$, $t = -q^{3J/2-1/4}x$ and $c = -1$ into the generating function (A.14) for the dual q-Kravchuk polynomials and employing the relation

$$\frac{(q^{-m};q)_n}{(q;q)_n} = (-1)^n q^{n(n-1)/2-mn} \begin{bmatrix} m \\ n \end{bmatrix}_q \tag{3.27}$$

results in the following expansion:

$$(q^{1/4-J/2}x;q)_{J-m}(-q^{1/4-J/2}x;q)_{J+m} = \sum_{n=0}^{2J} q^{n(n-J-1/2)/2} \begin{bmatrix} 2J \\ n \end{bmatrix}_q K_n(\lambda(J-m);-1,2J|q)x^n \tag{3.28}$$

where $\lambda(J-m) = q^{-J}(q^m - q^{-m})$. Taking into account the explicit form of the normalization constant $c_m^J(q)$ in (3.14), relation (3.28) is equivalent to the expansion

$$\varphi_m^J(x;q) = \sum_{m'=-J}^{J} \alpha_{m',m}^J(q) f_{m'}^J(x;q) \tag{3.29}$$

where the connection coefficients $\alpha_{m',m}^J(q)$ are equal to

$$\alpha_{m',m}^J(q) = q^{(J+m')(J+m'-1)/4} \begin{bmatrix} 2J \\ J+m' \end{bmatrix}_q^{1/2} K_{J+m'}(\lambda(J-m);-1,2J|q). \tag{3.30}$$

As in the classical case, the connection coefficients $\alpha_{m,m'}^J(q)$ are the particular case of the matrix elements $A_{m,m'}^J(q)$ of the intertwining operator (3.6) with $A_{-J,m'}^J(q) = 1$, $-J \leqslant m' \leqslant J$.

4. Concluding remarks

In section 3 we have explicitly constructed a 'nonstandard' basis for representations of the quantum algebra $su_q(2)$. The basis functions (3.21b) are common eigenfunctions of two difference operators: the Casimir operator (3.22) and the operator $\tilde{J}_1(q)$ of equation (3.5).

The relation between the standard and nonstandard basis functions is given by equation (3.29) and involves the dual q-Kravchuk polynomials. Both sets of basis functions are eigenfunctions of the Casimir operator. However, the additional operator, that actually determines the basis, is a differential operator $J_3(x; q) = J_3(x)$ (see (3.13)) for the standard basis, but a difference operator $\tilde{J}_1(q)$ for the nonstandard one. Thus overlap functions provide relations between solutions of differential and difference equations.

The representations were realized in a space of functions of one variable. It would be of interest to consider other realizations, for instance, functions on a two-dimensional sphere, as in [4]. An interesting question is whether the nonstandard basis corresponds to some type of separable coordinates, or whether the basis functions will be genuinely functions of two variables.

A more general question is that of bases for representations of higher-dimensional quantum algebras. They may have Lie subalgebras that can be realized by differential operators and complementary subspaces, realized by difference operators. This raises the possibility of having both standard bases, corresponding to subgroup chains, and nonstandard ones, corresponding to the diagonalization of commuting sets of difference operators.

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Appendix

1. The classical case. The Kravchuk polynomials are defined [17] as

$$K_n(x; p, N) := {}_2F_1(-n, -x; -N; 1/p) \qquad 0 < p < 1 \qquad n = 0, 1, \ldots, N \tag{A.1}$$

where

$$ {}_2F_1(-n, b; c; z) = \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

is the terminating hypergeometric series of Gauss and $(a)_k = \Gamma(a + k)/\Gamma(a)$ is the shifted factorial. These polynomials satisfy the three-term recurrence relation

$$\begin{aligned} p(N-n)K_{n+1}(x; p, N) &+ n(1-p)K_{n-1}(x; p, N) \\ &= [p(N-n) + n(1-p) - x]K_n(x; p, N) \end{aligned} \tag{A.2}$$

with the initial condition $K_0(x; p, N) = 1$. As follows from the definition (A.1), the Kravchuk polynomials are self-dual, i.e.

$$K_n(m; p, N) = K_m(n; p, N) \qquad m, n \in \{0, 1, \ldots, N\}. \tag{A.3}$$

Euler's transformation formula

$$ {}_2F_1(a, b; c; z) = (1-z)^{-b} {}_2F_1\left(c - a, b; c; \frac{z}{z-1}\right)$$

for the Gauss hypergeometric series ${}_2F_1$ yields another property of the Kravchuk polynomials

$$K_n(x; p, N) = \left(\frac{p-1}{p}\right)^n K_n(N - x; 1 - p, N). \tag{A.4}$$

The Kravchuk polynomials (A.1) have a generating function of the form

$$\left(1 - \frac{(1-p)}{p} t\right)^x (1+t)^{N-x} = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} K_n(x; p, N) t^n. \tag{A.5}$$

In the case when $t = x$, $x = J - m$, $N = 2J$ and $p = \frac{1}{2}$, the generating function (A.5) is

$$(1+x)^{J+m}(1-x)^{J-m} = \sum_{n=0}^{2J} \begin{bmatrix} 2J \\ n \end{bmatrix} K_n(J-m; \tfrac{1}{2}, 2J) x^n. \tag{A.5$'$}$$

2. The q-case. The q-Kravchuk polynomials $K_n(q^{-s}; p, N; q)$ on the q-linear lattice $x(s) = q^{-s}$ are defined in terms of the terminating basic hypergeometric series ${}_2\phi_1(q^{-n}, b; c; q; z)$ [18] as

$$\begin{aligned} K_n(q^{-s}; p, N; q) &= \frac{(q^{s-N}; q)_n}{(q^{-N}; q)_n} q^{-ns} {}_2\phi_1(q^{-n}, q^{-s}; q^{N-s-n+1}; q; -pq^{n+N+1}) \\ &= \frac{(q^{s-N}; q)_n}{(q^{-N}; q)_n} q^{-ns} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q^{-s}; q)_k}{(q^{N-s-n+1}; q)_k (q; q)_k} (-pq^{n+N+1})^k \end{aligned} \tag{A.6}$$

where $n = 0, 1, \ldots, N$. In the limit as $q \to 1^-$,

$$\lim_{q\to 1^-} K_n(q^{-s}; p, N; q) = K_n\left(s; \frac{1}{p+1}, N\right). \tag{A.7}$$

The dual q-Kravchuk polynomials $K_n(x(s); c, N|q)$ on the q-quadratic lattice $x(s) = q^{-s} + cq^{s-N}$ are given by [19]

$$K_n(x(s); c, N|q) = \frac{(q^{s-N}; q)_n}{(q^{-N}; q)_n} q^{-ns} {}_2\phi_1(q^{-n}, q^{-s}; q^{N-s-n+1}; q; cq^{s+1}). \tag{A.8}$$

These polynomials satisfy a three-term recurrence relation

$$\begin{aligned} (1 - q^{n-N}) K_{n+1}(x(s); c, N|q) + cq^{-N}(1-q^n) K_{n-1}(x(s); c, N|q) \\ = [x(s) - (1+c)q^{n-N}] K_n(x(s); c, N|q). \end{aligned} \tag{A.9}$$

In the limit as $q \to 1^-$,

$$\lim_{q\to 1^-} K_n(x(s); 1 - 1/p, N|q) = K_n(s; p, N). \tag{A.10}$$

For integer values of $s = m = 0, 1, 2, \ldots, N$ the dual q-Kravchuk polynomials (A.8) and the q-Kravchuk polynomials (A.6) are related in the following way:

$$K_n(x(m); c, N|q) = K_m(q^{-n}; -cq^{-N}, N; q). \tag{A.11}$$

This is a q-extension of the self-duality property (A.3) of the Kravchuk polynomials (A.1). A q-extension of (A.4) is

$$K_n(x(N-s); c, N|q) = c^n K_n(x(s); c^{-1}, N|q). \tag{A.12}$$

This property of the dual q-Kravchuk polynomials $K_n(x(s); c, N|q)$ follows from the transformation formula (see [15, p 16]):

$${}_2\phi_1(q^{-n}, b; c; q; z) = q^{-n(n+1)/2} \frac{(b; q)_n}{(c; q)_n} (-z)^n {}_2\phi_1(q^{-n}, q^{1-n}/c; q^{1-n}/b; q; cq^{n+1}/bz) \tag{A.13}$$

for the terminating basic hypergeometric series ${}_2\phi_1$.

Finally, a q-extension of the expansion (A.5) is a generating function for the dual q-Kravchuk polynomials (see [15, p 103]),

$$(q^{-N}t;q)_{N-k}(cq^{-N}t;q)_k = \sum_{n=0}^{N} \frac{(q^{-N};q)_n}{(q;q)_n} K_n(x(k);c,N|q)t^n \qquad \text{(A.14)}$$

where $0 \leqslant k \leqslant N$ and $x(k) = q^{-k} + cq^{k-N}$.

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